Last time : short exact sequences $1 \to K \xrightarrow{f} G \xrightarrow{g} L \to 1$

inj, surj

and $\mathrm{Im} f = \mathrm{Ker}\, g$

Big theorem : a given short exact sequence $1 \to K \underset{\phi}{\overset{f}{\rightleftarrows}} G \underset{\psi}{\overset{g}{\rightleftarrows}} L \to 1$ is

equivalent to $1 \to K \to K \times L \to L \to 1$ $\iff$ $\exists$ retraction $\phi \circ f = \mathrm{Id}_K$

$\Downarrow$ $\Downarrow$

equivalent to $1 \to K \to K \rtimes L \to L \to 1$ $\iff$ $\exists$ section $g \circ \psi = \mathrm{Id}_L$

for some action $L \overset{\Phi}{\to} K$

If we assume G abelian, then the vertical arrows are $\Updownarrow$ because a semidirect product of abelian groups is abelian if and only if it is the direct product (trivial action Φ)

Today : abelian group ($a+b$ instead of ab
na instead of a^n, $\forall n \in \mathbb{Z}$
0 instead of 1)

Def : We say that $g_1, \dots, g_k \in G$ (abelian) generate G if $\forall g \in G$ can be written as $a_1 g_1 + \dots + a_k g_k$, $a_i \in \mathbb{Z}$

(G is called finitely generated if $\exists$ such a finite set of generators)

Ex: $\mathbb{Z}$ is generated by 1 (also -1)

$\mathbb{Z}^r = \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{r \text{ times}} = \{ (n_1, \dots, n_r) \mid n_1, \dots, n_r \in \mathbb{Z} \}$ addition componentwise

is generated by $e_1 = (1, 0, 0, \dots, 0)$
$e_2 = (0, 1, 0, \dots, 0)$
$\vdots$
$e_r = (0, 0, 0, \dots, 1)$

$\forall$ finite abelian group is finitely generated

Non-Ex: $\mathbb{Q}$ (proof: suppose $\mathbb{Q}$ were fin. gen.; take a set of generators $\frac{b_1}{c_1}, \dots, \frac{b_k}{c_k}$ $b_i \in \mathbb{Z}, c_i \in \mathbb{N}$; any element of $\mathbb{Q}$ could be written as a linear combination $a_1 \frac{b_1}{c_1} + \dots + a_k \frac{b_k}{c_k}$ $a_i \in \mathbb{Z}$

$= \dfrac{\text{integer}}{c_1 \cdots c_k}$ $\cancel{}$)

Thm: any finitely generated abelian group G is iso to

$$G \cong \mathbb{Z}^r \times \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \dots \dots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$$

$r = \text{rank}(G) \geq 0$ prime power = elementary divisors of G

(the RHS is unique up to permuting the factors)

<u>Def</u>: (torsion) $\forall$ abelian group G, the function

(fix $n \in \mathbb{Z}$) $\quad$ $G \longrightarrow G$ is a homomorphism
$\qquad\qquad\qquad\qquad$ $g \rightsquigarrow ng$

Its kernel $= \{ g \in G \mid ng = 0 \} = \mathrm{Tors}_n(G)$ (n-th torsion subgroup)

<u>Ex</u>: $\mathrm{Tors}_0(G) = G$, $\mathrm{Tors}_1(G) = \{0\}$, $\mathrm{Tors}_{-n}(G) = \mathrm{Tors}_n(G)$

<u>Lemma</u>: $\forall$ abelian group G,

$$\mathrm{Tors}(G) = \bigcup_{n=1}^{\infty} \mathrm{Tors}_n(G)$$

set of torsion elements

is a subgroup of G.

<u>Proof</u>: show that $\mathrm{Tors}(G)$ is closed under

- identity, i.e. $0 \in \mathrm{Tors}(G)$; obvious b/c $0 \in \mathrm{Tors}_n(G)$ $\forall n$
- inverses, i.e. $g \in \mathrm{Tors}(G) \Rightarrow -g \in \mathrm{Tors}(G)$;
  holds b/c $g \in \mathrm{Tors}_n(G) \Rightarrow -g \in \mathrm{Tors}_n(G)$ $\forall n$
- addition : <u>Lemma</u>: $n \mid n'$ then $\mathrm{Tors}_n(G) \subseteq \mathrm{Tors}_{n'}(G)$
  $\qquad\qquad\qquad\qquad\qquad\qquad$ $ng = 0 \Rightarrow n'g = 0$

take $g \in \mathrm{Tors}_n(G) \subseteq \mathrm{Tors}_{mn}(G) \Rightarrow g + h \in \mathrm{Tors}_{mn}(G) \subset \mathrm{Tors}(G)$
$h \in \mathrm{Tors}_m(G) \subseteq$

$n \in Tors_m(G) \Rightarrow$

- $Tors_m(G) \cap Tors_n(G) = Tors_{gcd(m,n)}(G)$

$\supseteq$ consequence of boxed Lemma

proof of $\subseteq$: take $g \in Tors_m(G) \cap Tors_n(G)$

i.e. $mg = ng = 0 \Rightarrow gcd(m,n)g = \alpha m g + \beta n g = 0$

however, $gcd(m,n) = \alpha m + \beta n$ for some integers $\alpha, \beta \in \mathbb{Z}$

Cor: if $gcd(m,n) = 1$, then $Tors_m(G) \cap Tors_n(G) = \{0\}$.

Def: an abelian group G is called torsion-free if $Tors(G) = \{0\}$ (i.e. $ng = 0$ for $n > 0 \Rightarrow g = 0$)

Prop: $\forall$ abelian group G, $G/Tors(G)$ is torsion-free

Proof let $[g] \in G/Tors(G)$ be such that $0 = n[g] = [ng]$ $\Rightarrow ng \in Tors(G) \Rightarrow \exists\, m > 0$ s.t. $\underbrace{mn}_{>0} g = 0$

$\Rightarrow g \in Tors(G) \Rightarrow [g] = 0$

Ex: $\mathbb{Z}^r$ is torsion-free $\left(\text{if } m(n_1, \dots, n_r) = 0 \text{ for } m > 0 \right.$

$\Downarrow$

$(mn_1, \dots, mn_r) = 0 \Rightarrow mn_1 = \dots = mn_r = 0$

$\Downarrow$

$\left. (n_1, \dots, n_r) = 0 \Leftarrow n_1 = \dots = n_r = 0 \right)$

so is any subgroup of $\mathbb{Z}^r$

(more generally, it's easy to see that any subgroup of a torsion free group is torsion free)

Ex: a finite abelian group $G \neq \{0\}$ is never torsion-free

in fact, $G = \mathrm{Tors}(G)$ ($\Leftarrow$ elements have finite orders)

Ex: $G = \mathbb{Z}^r \times \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$

$$\mathrm{Tors}(G) = \{ (\underset{\mathbb{Z}}{n_1}, \ldots, n_r) \times (g_1, \ldots, g_k) \text{ s.t. } \exists\, m>0$$

$$\text{s.t. } (mn_1, \ldots, mn_r) \times (mg_1, \ldots, mg_k) = 0$$

$$\Updownarrow$$

$$mn_1 = \cdots = mn_r = 0 \ \& \ mg_1 = \cdots = mg_k = 0 \}$$

$$= \{ (0, \ldots, 0) \times (g_1, \ldots, g_k) \} \text{ are arbitrary}$$

$$= \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$$

$\Rightarrow$ $G/\mathrm{Tors}(G) \cong \mathbb{Z}^r$

(red implication follows from the general fact below:

if $G = G_1 \times G_2$, then $G/G_2 \cong G_1$ refers to normal subgroup $\{e\} \times G_2$)

Stepping Stone to Big Theorem

Thm: any finitely generated torsion free abelian group G is iso to

$$G \cong \mathbb{Z}^r \quad \text{for some } r \geq 0$$

Prop 1: any fin. gen. torsion free abelian group is $\cong$ a subgroup of $\mathbb{Z}^k$ for some $k \geq 0$

Prop 2: any subgroup of $\mathbb{Z}^k$ is $\cong \mathbb{Z}^r$ for some $r \geq 0$

Proof of Prop 1: $g_1, \dots, g_k \in G$ (abelian) are called linearly independent if $a_1 g_1 + \dots + a_k g_k = 0 \Rightarrow a_1 = \dots = a_k = 0$

they induce an injective homomorphism $\mathbb{Z}^k \xhookrightarrow{F} G$

$(a_1, \dots, a_k) \rightsquigarrow a_1 g_1 + \dots + a_k g_k$

Recall that G is finitely generated, so it has generators

$$g_1, \dots\dots, g_{k+l} \in G$$

assume $g_1, \dots, g_k$ are a maximal subset of linearly independent elements among the generators

$g_1, \dots, g_k, g_i$ are dependent, $\forall i > k$

i.e. $a_1 g_1 + \dots + a_k g_k = b_i g_i$ for some $b_i \in \mathbb{Z} \setminus 0$

$$\left\{ a_1 g_1 + \dots + a_k g_k \;\middle|\; a_i \in \mathbb{Z} \right\} \ni \begin{cases} b_{k+1} g_{k+1} \\ \vdots \\ b_{k+l} g_{k+l} \end{cases}$$

let $b = \text{l.c.m}(b_{k+1}, \dots, b_{k+l})$

$\Downarrow$

$$b g_{k+1}, \dots, b g_{k+l} \in \left\{ a_1 g_1 + \dots + a_k g_k \;\middle|\; a_i \in \mathbb{Z} \right\} = \mathrm{Im}\, F$$

$$b g_1, \dots, b g_k \in$$

$\Downarrow$

any linear combination of $b g_1, \dots, b g_{k+l} \in \mathrm{Im}\, F$

$\Downarrow$

$$\mathrm{Im}\, M = \{ bg \mid g \in G \} \subseteq \mathrm{Im}\, F$$

$$\begin{tikzcd} G \arrow[r, hook, "M"] \arrow[d, dashed, "\Upsilon"'] & G \\ \mathbb{Z}^k \arrow[ur, "F"'] & \end{tikzcd}$$

mult by b

injective because G is torsion free

$$\exists \; \Upsilon : G \hookrightarrow \mathbb{Z}^k \quad \text{s.t.} \quad M = F \circ \Upsilon \qquad \square$$

abelian

Lemma: any
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] & G \arrow[r, "g"] & \mathbb{Z}^l \arrow[r] \arrow[l, bend left, "\Psi"] & 0 \end{tikzcd}$$
is split, i.e. $\exists$ section Ψ

Proof: $\mathbb{Z}^l \ni e_1, \dots, e_l$; where should Ψ take the e_i's?

because $g \circ \Psi = \mathrm{Id}_{\mathbb{Z}^\ell}$, it better take each e_i to an element of $g^{-1}(e_i)$.

choose arbitrary $\left.\begin{array}{l} x_1 \in g^{-1}(e_1) \\ \vdots \\ x_\ell \in g^{-1}(e_\ell) \end{array}\right\} \in G$

$\Psi : \mathbb{Z}^\ell \longrightarrow G$

$\Psi((a_1, \dots, a_\ell)) = a_1 x_1 + \dots + a_\ell x_\ell$

is a homomorphism (easy)

$$\begin{array}{ccc} g \circ \Psi((a_1, \dots, a_\ell)) & = & g(a_1 x_1 + \dots + a_\ell x_\ell) \\ & & \| \\ (a_1, \dots, a_\ell) = a_1 e_1 + \dots + a_\ell e_\ell & = & a_1 g(x_1) + \dots + a_\ell g(x_\ell) \end{array} \Longrightarrow g \circ \Psi = \mathrm{Id}_{\mathbb{Z}^\ell}$$

<u>Proof of Prop 2</u> : (if $G \subseteq \mathbb{Z}^k$, then $G \cong \mathbb{Z}^r$ for some r)

induction on k : base case $k = 1$ $\Longrightarrow$ $G = \{0\} \cong \mathbb{Z}^0$ or $G = n\mathbb{Z} \cong \mathbb{Z}^1$

induction step (assume true for $G \subseteq \mathbb{Z}^{k-1}$, prove for $G \subseteq \mathbb{Z}^k$)

Consider the following short exact sequence

$(a_1, \dots, a_{k-1}) \rightsquigarrow (a_1, \dots, a_{k-1}, 0)$

$(b_1, \dots, b_{k-1}, b_k) \rightsquigarrow b_k$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{k-1} \arrow[r, "\iota"'] & \mathbb{Z}^k \arrow[r, "\pi"'] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "f"'] \arrow[u, hook] & G \arrow[r, "g"'] \arrow[u, hook] & L \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

(inclusion) (induced by π)

Define

$K = G \cap \operatorname{Im}\iota$ Why do f & g above form a s.e.s.?

$L = \tilde{\pi}(G)$

$$\begin{array}{ccc} \operatorname{Ker} g & \overset{?}{=\!=} & \operatorname{Im} f \\ \| & & \| \end{array}$$

$$\{x \in G \mid \tilde{\pi}(x) = 0\} = \{x \in G \mid x \in \operatorname{Im}\iota\} = G \cap \operatorname{Im}\iota$$

However, L is a subgroup of $\mathbb{Z}$, so we have two options

1. $L = 0 \implies G \subseteq \operatorname{Ker}\tilde{\pi} = \operatorname{Im}\iota \implies G$ is a subgroup of $\mathbb{Z}^{k-1} \implies$ ind. hyp gives $G \cong \mathbb{Z}^r$

2. $L = n\mathbb{Z} \cong \mathbb{Z}$, so the discussion above gives a s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & G \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, bend left, "\text{split by Lemma}"] & 0 \end{tikzcd}$$

$$\Downarrow$$

$$G \cong K \times \mathbb{Z}$$

but K is a subgroup of $\mathbb{Z}^{k-1}$, so the induction hypothesis gives us $K \cong \mathbb{Z}^r$ for some r $\Bigg\} \implies G \cong \mathbb{Z}^{r+1}$

$\square$

<u>Corollary</u>: $\forall$ finitely generated abelian group G

$$G \cong \mathbb{Z}^r \times \operatorname{Tors}(G)$$

next time $\cong \mathbb{Z}/p_1^{d_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_k^{d_k}\mathbb{Z}$

Proof: $G/\mathrm{Tors}(G)$ is a torsion free abelian group also clearly finitely generated

$$G/\mathrm{Tors}(G) \underset{\text{previous theorem}}{\cong} \mathbb{Z}^r$$

$\exists$ s.e.s.

$$0 \longrightarrow \mathrm{Tors}(G) \longrightarrow G \longrightarrow \mathbb{Z}^r \longrightarrow 0$$

Lemma

$$\Downarrow$$

$$G \cong \mathbb{Z}^r \times \mathrm{Tors}(G)$$

Please complete indicative evaluations